1 Self-affine surfaces

The most common example of a self-affine signal or surface is the trajectory of a one-
dimensional Brownian motion. We assume that the position of the Brownian particle
is $z(t)$ at a time $t$, and that the random walker started at a position $z = 0$ when
$t = 0$. If we now approach this motion from a statistical point of view, the fundamental
quantity to consider is the probability $p(z; t) \, dz$ of finding the particle in an interval
$[z - dz/2, z + dz/2]$ at a time $t$. The probability density $p(z; t)$ is easily found by solving
the diffusion equation

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial z^2} \quad , \tag{1}$$

where we have set the diffusion constant equal to unity for simplicity. If we have the
initial condition $p(z; 0) = \delta(z)$, the solution to the equation is the gaussian distribution

$$p(z; t) = \frac{e^{-z^2/2t}}{\sqrt{2\pi t}} \quad . \tag{2}$$

The gaussian distribution has the scaling property

$$\lambda^{1/2} \, p(\lambda^{1/2} z; \lambda t) = p(z; t) \quad . \tag{3}$$

The meaning of this scale invariance is that the Brownian particle spreads out in space
as the square root of time. In particular, the second moment of the position of the
particle at time $t$ is

$$\langle z^2 \rangle = \int_{-\infty}^{+\infty} dz \, z^2 p(z, t) = t \quad . \tag{4}$$

A self-affine function is characterized statistically in the same way as the Brownian particle, i.e., by the probability density $p(z, t)$. The property that defines self affinity is the scale invariance

$$\lambda^H \, p(\lambda^H z; \lambda t) = p(z; t) \quad . \tag{5}$$

The scaling exponent $H$ is the Hurst, or roughness exponent. In the case of the Brownian particle, $H = 1/2$. For the function to be called self affine, the Hurst exponent is in the
range $0 \leq H \leq 1$. The upper limit ensures that the function grows slower than unity
with time,

$$\frac{\sqrt{\langle z^2 \rangle}}{t} \propto t^{H - 1/2} \to 0 \quad , \tag{6}$$

as $t \to \infty$. When $z$ characterizes a surface, i.e., $z(t)$ is the height of the surface at point
t, The property (6) signifies that the surface is asymptotically flat. When $H < 0$, $\langle z^2 \rangle$
approaches a constant.

Consider now a general surface with a height profile given by a function $h(x)$. In general,
the full distribution function of the surface height fluctuations $p(z = h(x + \delta) - h(x))$
is not always accessible due to the limited statistics. It is therefore common to only
consider the structure function corresponding to the root of the 2nd moment of the
increment $z_x(\delta) = h(x + \delta) - h(x)$ on a scale $\delta$:

$$C_2(\delta) = \langle |h(x + \delta) - h(x)|^2 \rangle^{1/2} \sim \delta^H. \quad (7)$$

The average is taken over the spatial coordinate $x$.

For the one-dimensional Brownian motion, the statistical distribution of increments $\Delta h$ is a Gaussian:

$$P(\Delta h) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\Delta h \right)^2 / 2\sigma^2}, \quad (8)$$

Note that the self-affinity enters through the variance of the distribution $\sigma^2 \propto \delta^{2H}$, where $H = 1/2$. Moreover, the moments Eq. (7) are easily calculated,

$$C_q(\delta) = \sqrt{2} \left( \Gamma((q + 1)/2) / \sqrt{\pi} \right)^{1/q} \delta^H.$$

**Problem 1:** Verify the calculation of the moments.

It is important (!) to pay attention to the fact that the scaling exponent

$$C_q(\delta) \sim \delta^H \quad (9)$$

is independent of $q$. In general this will almost always be the case. That being said, interfaces do exist for which this is not the case\textsuperscript{1}

\textsuperscript{1}Multi-affine surfaces have e.g. been discussed in S. Santucci, Phys. Rev. E \textbf{75}, 016104 (2007)
2 Flow in a Hele-Shaw Cell

2.1 Stokes flow

The Navier-Stokes equation for an incompressible, isotropic and homogeneous fluid when
the gravitational field is not important has the form

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_0} \nabla p + \frac{\mu}{\rho_0} \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0 \]  

(10)

where \( \rho_0 \) is the fluid density and \( \mu \) is the dynamic viscosity.

We now consider a steady flow of the fluid with a velocity which is assumed to be of
the order of magnitude of \( |\mathbf{u}| \approx U \). Moreover, it is assumed that the velocity does not
change more than \( U \) in magnitude over a region of size \( L \). We then have \( |\nabla \mathbf{u}| \approx U/L \).
Finally it is assumed that \( |\nabla^2 \mathbf{u}| \approx U/L^2 \). We then have that the Reynolds number \( Re \)
is given by

\[ Re \approx \frac{|(\mathbf{u} \cdot \nabla) \mathbf{u}|}{|\frac{\mu}{\rho_0} \nabla^2 \mathbf{u}|} \approx \frac{\rho_0 U L}{\mu}. \]

For a steady flow we have that \( \partial \mathbf{u} / \partial t \approx 0 \) and therefore in the limit of \( Re \ll 1 \), the full
Navier-Stokes equation reduces to the Stokes equation

\[ \nabla^2 \mathbf{u} - \frac{1}{\mu} \nabla p = 0. \]  

(11)

The flow is still assumed to satisfy the incompressibility condition,

\[ \nabla \cdot \mathbf{u} = 0. \]  

(12)

If we apply the operator \( \nabla \cdot \) on both sides of Eq. (11) we have by the incompressibility
that the pressure satisfies the a Laplace equation

\[ \nabla^2 p = 0 \]  

(13)

2.2 Darcy’s flow law

A Hele-Shaw cell is a common experimental system in which a fluid is trapped in an
infinitesimal small gap between two parallel flat plates. The cell is assumed to be located
in the x-y plane and to be of thickness \( b \) in the z-direction. We shall now consider
an experiment where the cell contains two immiscible fluids separated by an interface
located at a position \( y = h(x,t) \). One fluid is located at \( y < h(x,t) \) and the other at
\( y > h(x,t) \), see Fig. 1.

If the thickness \( b \) is sufficiently small, one can derive a set of approximate equations for
the flow in the Hele-Shaw cell. We do that by averaging the flow inside the gap of the
cell. The averaged incompressibility equation becomes on the form

\[ \frac{1}{b} \int_0^b \nabla \cdot \mathbf{u} \, dz = \partial_x \tilde{u}_x + \partial_y \tilde{u}_y + (u_z|_{z=b} - u_z|_{z=0}) = \partial_x \tilde{u}_x + \partial_y \tilde{u}_y. \]  

(14)
Figure 1: Hele-Shaw cell made from two plates separated by a gap of size $b$. The fluid 1 is injected into fluid 2 from the left.

We have here used the assumption that the velocity of the fluid vanishes at the boundaries of the cell. Since all the velocity components vanish at the boundaries, we expect that the maximum flow rate is attained in the middle of the gap between the two plates. It is therefore assumed that the velocity components can be approximated by a parabolic shape in the $z$-coordinate:

$$u(x, y, z) = -v(x, y)z(z - b)$$  \hspace{1cm} (15)

where $v(x, y)$ is a gap averaged strength.

In general the change in fluid flow velocity is predominant in the $z$-direction, where it changes over an infinitesimal thickness from zero at the boundaries to a maximal flow between the plates. Therefore the Laplacian term in Eq. (11) is almost entirely given by the second order derivate with respect to $z$ alone i.e. $\nabla^2 u \approx \partial_z^2 u$. It then follows by inserting Eq. (15) in Eq. (11) that

$$v = -\frac{1}{2\mu} \nabla p.$$  \hspace{1cm} (16)

Performing a gap average of the velocity component $u_j$ we end up with Darcy’s law

$$\tilde{u}_j = \frac{1}{b} \int_0^b u_j(x, y, z) \, dz = \frac{v_j b^2}{6} = -\frac{b^2}{12\mu} \partial_j p$$  \hspace{1cm} (17)

In the gap averaged quantities we therefore end up with the following set of equations (the three dimensional problem is now reduced to a two dimensional problem of a flow in a plane)

$$\nabla^2 p = 0, \quad \tilde{u} = -\frac{b^2}{12\mu} \nabla p$$  \hspace{1cm} (18)

### 2.2.1 Example: solution to the pressure field for a flat interface

We now consider the case where the fluids move with a steady velocity and form an interface which at a time $t$ is located at $y = h(x, t) = V_0 t$, i.e. the fluids are driven by an appropriate external pressure gradient applied at the remote boundaries $y \to \pm \infty$. 
Figure 2: Experimental setup where air is injected into a viscous fluid from a small aperture in the middle of a Hele-Shaw cell. As the viscous fluid is displaced by the air a characteristic ramified pattern emerge.

In both fluids Eq. (18) must be satisfied, i.e.

\[ \nabla^2 p_1 = 0 \quad \text{and} \quad \nabla^2 p_2 = 0. \]  

(19)

Moreover, the interface must stay coherent, that is the normal velocities in both fluids when approaching the interface are identical. In addition, the pressure is continuous across the interface\(^2\) i.e.

\[ V_0 = -\frac{b^2}{12\mu_1} (n \cdot \nabla p_1)|_{y \to h^-} = -\frac{b^2}{12\mu_2} (n \cdot \nabla p_2)|_{y \to h^+} \quad \text{and} \quad p_1|_{y \to h^-} = p_2|_{y \to h^+}. \]  

(20)

Here we have considered a normal vector pointing in the same direction for both fluids at the interface. For a flat interface separating the two fluids the solution is independent of \(x\) and therefore from the above conditions we have that

\[ p_1(x, y, t) = -\frac{12\mu_1 V_0}{b^2} y + k_1(t), \quad \text{and} \quad p_2(x, y, t) = -\frac{12\mu_2 V_0}{b^2} y + k_2(t), \]  

(21)

where \(k_i(t)\) are integration constants that depend on time.

**Problem 2:** Calculate the pressure fields given in Eq. (21) using Eqs. (19) and (20). Discuss the possible forms of \(k_1(t)\) and \(k_2(t)\), how do they depend on \(t\)?

\(^2\)That is, there is no surface tension at the interface
2.3 Saffman-Taylor instability

2.3.1 Linear stability analysis

We shall now find the solution to the Laplace equation when the flat surface is perturbed by a small amplitude function $\epsilon(t)h(x)$ where $h(x, t) = V_0 t + \epsilon(t)h(x)$ and $\epsilon(t) \ll 1$, that is, the new field satisfying the Laplace equation is going to be written in terms of an expansion around the solution to the flat interface. Formally the solution is written as

$$p(x, y) = p^{(0)}(x, y) + \epsilon(t)p^{(1)}(x, y) + O(\epsilon(t)^2),$$

(22)

Evaluated at a point on the interface $y = V_0 t + \epsilon h(x)$, we can expand this expression to linear order in $\epsilon$ (we omit the argument $t$ of $\epsilon(t)$ to make the following expressions more readable)

$$p(x, V_0 t + \epsilon h(x)) = p^{(0)}(x, V_0 t) + \epsilon h(x) \frac{\partial}{\partial y} p^{(0)}(x, y)|_{y=V_0 t} + \epsilon p^{(1)}(x, V_0 t) + O(\epsilon^2).$$

(23)

Zero order terms

For a flat interface separating the two phases the equation for the harmonic potential or the pressure is given by the translational invariant (in the x-direction) solution computed above

$$p_i^{(0)}(x, y, t) = -\frac{12\mu_i V_0}{b^2} y + k_i(t)$$

(24)

First order terms

The first order corrections are determined by Fourier transforming the perturbation, $h(x) = \int dk \tilde{h}(k)e^{ikx}$ and $p^{(1)}(x, y) = \int dk \tilde{p}_i^{(1)}(k, y)e^{ikx}$. If we Fourier transform the Laplace equation we end up with an equation on the form

$$(-k^2 + \partial_y^2) \tilde{p}_i^{(1)}(k, y, t) = 0$$

(25)

which has solutions for the two fluids on the form

$$\tilde{p}_1^{(1)}(k, y) = A_1(k)e^{ky-V_0 t} \quad \text{and} \quad \tilde{p}_2^{(1)}(k, y) = A_2(k)e^{-ky-V_0 t}$$

(26)

where we require that the pressure due to the perturbation decays away from the interface, i.e. the sign of the exponentials is determined and the time dependence enters into the exponent.

The combined solutions of zero and first order terms are given by

$$\tilde{p}_1(k, y, t) = -\frac{12\mu_1 V_0}{b^2} y + k_1(t) + \epsilon A_1(k)e^{ky-V_0 t} + O(\epsilon^2)$$

(27)

and

$$\tilde{p}_2(k, y, t) = -\frac{12\mu_2 V_0}{b^2} y + k_2(t) + \epsilon A_2(k)e^{-ky-V_0 t} + O(\epsilon^2)$$

(28)

Alternatively, one could assume a general form where e.g. $\tilde{p}_1^{(1)}(k, y) = A_1(k, t)e^{ky}$. Then from matching the boundary conditions one would end up with the same time dependence in the exponent.
2.3 Saffman-Taylor instability

The growth of the amplitude of the perturbation is given by the growth in the normal direction of the surface where the surface has a normal \( (n_x, n_y) = (-\epsilon h', 1)/\sqrt{1 + (\epsilon h')^2} \) and a tangent \( (t_x, t_y) = (1, \epsilon h')/\sqrt{1 + (\epsilon h')^2} \) vector, respectively. Note that we have contributions to the surface growth from both the zero and first order terms. The normal velocity of the interface is given by an expression

\[
V_n = n_x v_x + n_y v_y = v_y = \partial_t h(x, t) = V_0 + h(x)\partial_t \epsilon(t). \tag{29}
\]

Note that \( n_x v_x \) is at least \( \mathcal{O}(\epsilon^2) \) since \( n_x \) is \( \mathcal{O}(\epsilon) \) and \( v_x \) has no zero order term since there is no \( x \)-dependence in Eq. (24). Similarly the normal velocity follows from Darcy’s law,

\[
V_n = -\frac{b^2}{12\mu_1} (n \cdot \nabla p_i) = V_0 - \frac{b^2}{12\mu_1} \partial_y p_1^{(1)} + \mathcal{O}(\epsilon^2). \tag{30}
\]

If we now assume that \( \epsilon(t) = \epsilon_0 \exp(\omega t) \) and use Darcy’s law we have for the first order terms that (we have here compared the zero and first order terms and do only show the first order equation)

\[
-\frac{b^2}{12\mu_1} \partial_y p_1^{(1)}(k, y)|_{y=V_0 t} = -\frac{b^2}{12\mu_1} k A_1(k) = \omega \tilde{h}(k) \tag{31}
\]

and

\[
-\frac{b^2}{12\mu_1} \partial_y p_2^{(1)}(k, y)|_{y=V_0 t} = \frac{b^2}{12\mu_2} k A_2(k) = \omega \tilde{h}(k) \tag{32}
\]

In both these equations the latter equality sign follows from the first order term in the right hand expression of Eq. (29). At the same time we require continuity of the pressure field across the interface, which must be valid for both the zero and first order terms independently. For the first order terms this means using Eq. (23) that

\[
\frac{12\mu_1 V_0}{b^2} \tilde{h}(k) + A_1(k) = \frac{12\mu_2 V_0}{b^2} \tilde{h}(k) + A_2(k) \tag{33}
\]

If we now rearrange terms and use Eqs. (31) and (32) we have the dispersion relation

\[
\omega = kV_0 \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \tag{34}
\]

**Problem 3:** Perform all the calculational steps leading to the final dispersion relation Eq. (34).

**Problem 4:** Is the interface between two immiscible fluids stable when a more viscous fluid is displacing a less viscous fluid? Draw a stability diagram showing regions as function of the viscosity of the two fluid where the interface is stable and unstable, respectively (consider different velocities \( V_0 \)). What would the pattern in Fig. 2 look like, if the same fluid was injected into air?
3 Fractals

3.1 Box-counting dimension

The dimension of a fractal set is found from covering it with boxes of size $\epsilon$ and count the number of boxes, $N(\epsilon)$, needed to cover the set. If this number follows a power law in $\epsilon$ with an exponent $-D$, we shall call $D$ the box-counting dimension or just the box-dimension

$$N(\epsilon) \sim \epsilon^{-D}$$  \hfill (35)

where the symbol $\sim$ is used for scaling relations that hold for small values of $\epsilon$. In other words, the box dimension follows from taking the limit

$$D = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{-\log \epsilon}$$  \hfill (36)

Note that in mathematics or in more rigorous presentations, it is common to encounter the Hausdorff fractal dimension. The Hausdorff dimension is for many well behaved fractals identical to the box-dimension, however for certain sets, such as the set of rational numbers, the box-dimension will obviously be $D_{box} = 1$ whereas $D_{Haus} = 0$. Here, we shall only consider the box-dimension.

3.2 Fractals generated by similarity transformations

Similarity transformations $S_i$ are transformations with fixed scaling factors $\{\ell_i\}$ such that

$$|S_i(x) - S_i(y)| = \ell_i|x - y|$$

Let $F$ denote a fractal set which is invariant under a given set of similarity transformations $\{S_i\}_{i=1}^n$, i.e.

$$F = \bigcup_{i=1}^n S_i(F)$$

If we assume that the sets $S_i(F)$ are disjoint (except possible intersections of the boundaries of the sets) and the scaling factors all satisfy $\ell_i \leq 1$. The fractal dimension (similarity dimension) $D$ then follows from the relation

$$\sum_i \ell_i^D = 1$$

3.3 Example: Two-scale Cantor set

The two-scale Cantor set is generated from two similarity transformations defined on the unit interval $I = [0, 1]$ and with scaling factors $\ell_1$ and $\ell_2$.

$$S_1(x) = \ell_1x \quad S_2(x) = \ell_2x + \alpha$$  \hfill (37)
where \( \alpha \geq \ell_1 \) and \( \ell_2 + \alpha \geq 1 \).

The number of boxes of size \( \epsilon \) needed to cover the fractal can be written recursively on the form (convince yourself that this is the case)

\[
N(\epsilon) = N(\epsilon/\ell_1) + N(\epsilon/\ell_2)
\] (38)

If we now divide with \( N(\epsilon) \) on both sides and take the limit of \( \epsilon \to 0 \), we end up with an equation for the dimension

\[
1 = \ell_1^D + \ell_2^D
\] (39)

**Problem 5:**

A) Find the similarity transformations and dimension of the fractal generated from the generator in the figure below using an appropriate choice of the scaling parameters (note that you may have to solve the equation for the dimension numerically)

B) The easiest way to generate the fractal is to apply the similarity transformations iteratively to a single point \( x \in F \). That is, your fractal will numerically consist of the points

\[
x, S_{i_1}(x), S_{i_2} \circ S_{i_1}(x), S_{i_3} \circ S_{i_2} \circ S_{i_1}(x), \ldots
\]

The indices \( i_k \) denote a random selection of one of the similarity transformations. Use this scheme to generate the fractal from A).

C) Compute numerically the box-dimension of your fractal and compare the result with that found in A).
4 Multifractals

4.1 Generalized dimensions

Geometrical objects with a scale invariant structure are often classified by a single exponent - a fractal dimension. Consider now measures $\mu$ such as a mass density, electric charge or probability density distributed on a fractal set. Assume that the distribution is highly non-uniform and even has a singularities (such as the electric field close to the tip of a charged needle). In such cases the fractal dimension provides limited or no information about the distribution of the measure.

Analogously to the box-counting method, we shall now cover the measure distributed on a fractal set by a hyper-cubic lattice of boxes of linear size $\epsilon$. Inside each box we integrate up the measure $\mu$,

$$p_i = \int_{\text{i'th box}} d\mu(x),$$

where $p_i$ is the weight of each box and acts like a coarse grained measure. Boxes with zero weight are disregarded. From the set of weights, a moment sum is now defined as the sum

$$\chi_q(\epsilon) \equiv \sum_i p_i^q$$

for all real values of $q$. We see that for $q = 0$ the sum equals the number of boxes $N(\epsilon)$ covering the fractal and therefore $\chi_0(\epsilon)$ scales with the box-dimension of the fractal, i.e. $\chi_0(\epsilon) \sim \epsilon^{-D}$.

Let us now assume that all boxes $p_i$ have the same weight, the moment sum $\chi_q(\epsilon)$ then becomes

$$\sum_i P_i^q = N(\epsilon) \frac{1}{N(\epsilon)^q} = \epsilon^{(q-1)D}$$

That is, it scales with an exponent increases linearly with the value of $q$. This is analogously to the scaling of the structure function calculated for the self-affine interfaces. In general, non-trivial measures will not be linear in $q$ and a general scaling exponent is introduced

$$\chi_q(\epsilon) \sim \epsilon^{\tau(q)}$$

Note that $\chi_q$ is a decreasing function in $q$ therefore $\tau(q)$ must be monotonically increasing. Based on the example with uniform weights, a generalized fractal dimension $D(q)$ is now defined from the relation

$$\tau(q) = (q - 1)D_q$$

It readily follows from this definition that the generalized dimension $D_0$ equals the dimension of the fractal $D_0 = D$, and the factor of $q - 1$ guarantees, in the case where
$q = 1$, that the weights in (40) sum to unity $(\epsilon^{(1-1)D_1})$ for all values of $\epsilon$. In terms of the boxes covering the support of the measure, the generalized dimensions can be calculated from taking the limit

$$D_q = \lim_{\epsilon \to 0} \frac{1}{q-1} \log \frac{\chi_q(\epsilon)}{\log \epsilon} \quad (45)$$

A simple application of Hölder's inequality shows that $D_q$ generally is a monotonically decreasing function in $q$.

**Problem 6: Information Dimension**

The dimension $D_1$ is often called the information dimension. Taking the limit of $q \to 1$ in Eq. (45) can you explain why that name is used?

### 4.2 Example: Iterated function scheme

Consider the four similarity transformations $S_0, S_1, S_2$, and $S_3$ of the unit square shown in figure 3. Each one rescales the square by a factor $l_j = \frac{1}{2}$ and performs a simple translation. The mappings together map one-to-one the square to the square. We now introduce a measure on the square in the following way. Choose an initial point $x$ randomly from inside the square. The point is now mapped by one of the transformations $S_j$, selected randomly, to a new point $x_{(j)} = S_j(x)$. We now continue this process iteratively by choosing one of the four maps randomly and form thereby form a sequence of points

$$x, x_{(j_1)}, x_{(j_1, j_2)}, x_{(j_1, j_2, j_3)}, \cdots \quad (46)$$

Here we use the notation $x_{(j_1, j_2)} = S_{j_2}(x_{j_1}), x_{(j_1, j_2, j_3)} = S_{j_3}(x_{j_1, j_2})$ etc. We may now choose to apply the individual transformations with different frequencies, i.e. in each iterative step we pick one of the transformations using a non-uniform probability distribution. If we now cover the initial square with small boxes, a measure is introduced by counting the fraction of points from the iterative sequence ending up in each box. That is in the limit of a long sequence with length $N$ the weight of each box is given by

$$p_i = \lim_{N \to \infty} \frac{N_i}{N}, \quad (47)$$

Where $N_i$ denotes the number of points inside the $i$'th box.

#### 4.2.1 Digression

If each similarity transformation is applied with a non-vanishing frequency, the sequence of points is easily shown to be dense in the unit square, since, using an appropriate sequence of transformations, the unit square can be mapped to any square with a diagonal spanned by points on the form $(i_x, i_y)2^{-n}$ and $(i_x + 1, i_y + 1)2^{-n}$. The value of $n$ can be chosen arbitrarily large and $i_x, i_y < 2^n$. A random point in the unit square will therefore by a finite, but sufficiently long, sequence of transformations come arbitrarily close to
any other point. Moreover, since the sequence is dense, it has the box dimension of the square, \( D = 2 \).

It is easy to show that if the similarity transformations are applied with the same frequency the iterative sequence will visit all regions with the same probability, thus the measure is uniform throughout the square.

If the transformations \( S_j \) are applied with non-uniform frequencies, \( A_j \), the measure will no longer be uniform. Even a minor change in frequencies will lead to large fluctuations in the density of points throughout the square as can be observed in figure 4, where two sequences are shown of length \( n = 100000 \) and the corresponding distributions of points.

For this example, equation (43) can be evaluated analytically. Assume that we have composed an infinite sequence of the four transformations by a random selection according to the probabilities \( A_j \). That is the sequence would have a fraction \( A_j \) of the transformation \( S_j \). Each time we apply the transformation \( S_j \) we end up in the corresponding region \( j \), thus the region \( j \) will have a fraction \( A_j \) of points from the sequence. If we now look at the \( N = 4^2 \) boxes for \( \epsilon = 2^{-2} \) we see that each of these boxes corresponds to a mapping of the unit square composed of two transformations \( S_{j_1} \circ S_{j_2} \). For instance will the square with the diagonal bounded by \((2,2)2^{-2}\) and \((3,3)2^{-2}\) follow from \( S_1 \circ S_2 \). In our infinite sequence of transformations it happens a fraction \( A_{j_1} \cdot A_{j_2} \) of times that we run into the subsequence \( S_{j_1} \circ S_{j_2} \) of mappings, hence end up in the corresponding box of width \( \epsilon = 2^{-2} \). Generally for a given \( \epsilon = 2^{-n} \) we have that all possible \( n \)-products of the probabilities \( A_j \) will describe the weights of all the squares of size \( \epsilon = 2^{-n} \). The moment sum therefore might be written using the multinomial expansion,

\[
\chi_q(2^{-n}) = \sum_{k_1+k_2+k_3+k_4=n} \frac{n!}{k_1! \cdots k_4!} (A_1^{k_1} \cdots A_4^{k_4})^q = (A_1^q + \cdots + A_4^q)^n \quad (48)
\]

If we compare this expression with (43) the coefficient \( \tau(q) \) follows directly. We shall now
Figure 4: Both figures are based on sequences of length \( N = 100000 \). On the left we have used the probabilities \((1/10, 3/10, 3/10, 3/10)\) and on the right \((4/10, 2/10, 3/10, 1/10)\). These plots give an idea of how the density distribution will look like in the limiting case, and they accentuate the presence of an underlying mask of interwoven fractals.

introduce an alternative approach for extracting the scaling exponent. The approach also applies to cases where the self-similarity transformations have different scales \( l_j \).

When we cover one of the four subregions \( j \) with boxes of width \( \epsilon \), we could as well by the self-similarity cover the whole unit square with boxes of width \( \epsilon/l_j \), where the weights of the boxes over the subregion, however, is a factor \( A_j \) lesser. We are therefore led to the expression

\[
\chi_q(\epsilon) = \sum_j A_j^q \chi_q(\epsilon/l_j)
\]

\[
\sim \sum_j A_j^q (\epsilon/l_j)^{\tau(q)}
\]

(49)

where we have used that \( \chi_q(\epsilon/l_j) \) scales with \( (\epsilon/l_j)^{\tau(q)} \). If we now, as well, replace \( \chi_q(\epsilon) \) on the right hand side with its scaling form and divide it out on both sides we end up with the expression

\[
1 = \sum_j \frac{A_j^q}{l_j^{\tau(q)}}
\]

(50)

This expression is very useful for determining \( \tau(q) \), since it is the power in the dominator that makes the sum equal unity. We have not written out the scaling coefficients \( l_j \) for
the above example explicitly, since the formula also applies to cases where they are not equal. When the scaling factors are equal, \( l_j = \frac{1}{2} \) for the present square example, the partition function can be written in a more simple form

\[
2^{\tau(q)}(A_1^q + A_2^q + A_3^q + A_4^q) = 1
\]

where the following expression determines \( \tau(q) \)

\[
\tau(q) = \frac{\log(A_1^q + A_2^q + A_3^q + A_4^q)}{-\log 2}
\]

We see that in the limit of large positive \( q \) the maximum probability will be dominating the right hand side whereas the minimum will be dominating for "large" negative \( q \), we therefore have that

\[
\tau(q) \sim q \frac{\log(\max_j A_j)}{-\log 2} \quad q \to \infty
\]

\[
\tau(q) \sim q \frac{\log(\min_j A_j)}{-\log 2} \quad q \to -\infty
\]

i.e. \( \tau(q) \) asymptotically is linear in the two limits considered. Furthermore by comparison with expression (44) we see that \( D_q \) approaches a constant value in the two limits, where e.g. \( D_{-\infty} = \log(\min_j A_j)/\log 2 \). Figure 5 shows how \( D_q \) varies as function of \( q \) for the two examples introduced in figure 4. Note that the minimum probability associated with a transformation is the same in the two examples, thus the two curves for \( D_q \) become equal in the limit of large negative \( q \)'s.
5 Multifractal Spectrum

In the previous section, we have introduced the two quantities $q$ and $\tau(q)$. However, within the theory of multifractals one often introduces a new pair of quantities $(\alpha, f(\alpha))$ to further describe the measure.

We introduce a local scaling exponent (Hölder exponent) for the measure in question by assuming that the probabilities introduced in (40) satisfy a scaling relation,

$$p_i(\epsilon) \sim \epsilon^{\alpha}$$

(54)

When we consider measures of fractal character, as those introduced in figure 4, we will often see that $\alpha$ assumes a broad range of values as different regions of the structure is considered. In cases where the measure is uniform, however, we have that $\alpha$ only assumes one value, the dimension of the support of the measure, e.g. $\alpha = 2$ for the square example above.

Typically there will be many boxes which scale with identical values of $\alpha$, $\alpha$ is here considered at a coarse grained level. In particular it turns out, in many examples, that the number of boxes $N_\alpha(\epsilon)$ associated with a given $\alpha$ scales with $\epsilon$ in a non-trivial manner,

$$N_\alpha(\epsilon) \sim \epsilon^{-f(\alpha)}$$

(55)

where usually $f(\alpha)$ is a smooth, positive and concave function of $\alpha$, i.e. it has a shape like $\cap$. We assume from now on that $f(\alpha)$ behaves so. By comparison with the general scaling equation involving fractal dimensions, we see that $f(\alpha)$ might be interpreted as a fractal dimension of the subset upon which all points have the same Hölder exponent $\alpha$. The measure, using this interpretation, therefore is the union of subsets, intervoven fractals, with a local scaling factor $\alpha$. The picture of intervoven fractals have given rise to the name multifractal and the function $f(\alpha)$ is referred to as the multifractal spectrum $f(\alpha)$.

The two expressions Eqs. (54) and (55) can now be used to rewrite the moment sum. In that way, we can establish a connection between the multifractal specturm and the generalised dimensions or $\tau(q)$,

$$\chi_q(\epsilon) = \sum_i p_i^q$$

$$\sim \sum_\alpha N_\alpha \epsilon^{\alpha q}$$

$$\sim \sum_\alpha \epsilon^{\alpha q - f(\alpha)}$$

(56)

In the limit of small $\epsilon$, the sum is dominated by the smallest exponent and therefore we have by comparison with equation (43) that

$$\tau(q) = \min_\alpha (\alpha q - f(\alpha))$$

(57)
The value minimizing the equation varies with $q$ and we write it as $\alpha_q$. In terms of $\alpha_q$ the above equation assumes the form $\tau(q) = q \alpha_q - f(\alpha_q)$. The summation over $\alpha$, expression (56), might, in the limit where the coarse graining of $\alpha$ disappears, be written as an integral over $\alpha$ which then is considered continuous. Therefore at the value $\alpha_q$, minimizing expression (57), we have an extremal point and the derivative in $\alpha$ vanishes, i.e.

$$\frac{d}{d\alpha}(\alpha q - f(\alpha))\bigg|_{\alpha_q} = 0 \quad (58)$$

The differentiation gives the relation

$$q = \frac{df}{d\alpha}(\alpha_q) \quad (59)$$

That is, the slope of $f$ equals $q$ at $\alpha_q$ and therefore for large positive and negative $q$’s we will see that the multifractal spectrum becomes very steep. If we now differentiate $\tau$ with respect to $q$ we end up with the important relation

$$\frac{d\tau}{dq} = \alpha_q + q \frac{d\alpha_q}{dq} - \frac{df}{d\alpha_q} \frac{d\alpha_q}{dq} = \alpha_q \quad (60)$$

Summarizing the above consideration we find that the pair $(q, \tau(q))$ is related to the new one $(\alpha, f(\alpha))$ through the Legendre transformation

$$\alpha_q = \frac{d\tau}{dq}(q) \quad , \quad f(\alpha_q) = \alpha_q q - \tau(q) \quad (61)$$

These equations are very useful whenever we want to find the multifractal spectrum from the generalized dimensions. Moreover, we note the following characteristics of the multifractal spectrum. From equation (59) we see that $f(\alpha_q)$ assumes its maximum for $q = 0$ and from equation of (61) we see that $f(\alpha_0) = -\tau(0) = D$, hence the maximal value of the multifractal spectrum equals the dimension of the support of the measure - no subset will have a dimension that is larger than the set it is part of. The multifractal spectrum is therefore bounded (it is positive and has a maximum value). Considering the limiting behavior of $q$ we establish the following relations, using the boundedness of $f(\alpha)$ and once more the latter part of equation (61),

$$\alpha_{\text{min}} = D_\infty = \lim_{\epsilon \to 0} \frac{\log p_{\text{max}}}{\log \epsilon}$$

$$\alpha_{\text{max}} = D_{-\infty} = \lim_{\epsilon \to 0} \frac{\log p_{\text{min}}}{\log \epsilon}$$

We therefore see that the $D_\infty$ and $D_{-\infty}$ define the left and right horizontal bounds of the spectrum, respectively. The righthand part of figure 5 shows the $f$-$\alpha$-spectra corresponding to the measures introduced in figure 4. The right branch of both spectra tend to zero as we approach the value of $\alpha_{\text{max}} = \log(10)/\log(2) \approx 3.32$ (see the text of figure 4), this is due to the fact, that the squares of width $\epsilon = 2^{-n}$, having the least weight are found by repeated use of the transformation(s) with the least probability. If there is one such transformation only, there will only be one square having the least
weight, and in the limit it will be truncated to a single point - a subset of dimension zero. The left branch of the $f$-$\alpha$ spectrum corresponding to the measure introduced on the lefthand in figure 4, however, does not tend to zero, this is because there are three transformations, which have the same maximum probability, giving rise to the fractal generated as follows,

\[ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \ldots \]

The dimension of this fractal follows from (36) considering the evolution of the above sequence,

\[ N_{2^{-n}} = 3^n = 2^n \left( \frac{\log 3}{\log 2} \right) = (2^{-n}) \left( \frac{\log 3}{\log 2} \right) \]

We see that

\[ D = \frac{\log 3}{\log 2} \approx 1.58, \]

in agreement with the minimum value of $f(\alpha)$ seen on the left branch of the non-dotted line in figure 5.

\begin{center}
\begin{tabular}{|c|}
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\textbf{Problem 7:} \\
Consider the similarity transformations $S_1(x) = 1/2x$ and $S_2(x) = 1/2x + 1/2$ on the unit interval [0, 1]. \\
A) What is the fractal dimension of the invariant set generated from these transformations? \\
B) Assume now that the transformations have weights $A_1 = 1/3$ and $A_2 = 2/3$. Find the spectrum of generalized dimensions $D_q$ and the multifractal spectrum $f(\alpha)$ of the density distribution generated from these two transformations. Discuss the values of $\alpha_{\min}$ and $\alpha_{\max}$. \\
\hline
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\end{center}